FRACTAL DIMENSIONS OF ROUGH DIFFERENTIAL EQUATIONS DRIVEN BY FRACTIONAL BROWNIAN MOTIONS

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ABSTRACT. In this work we study fractal properties of a d-dimensional rough differential equation driven by fractional Brownian motions with Hurst parameter $H > \frac{1}{4}$. In particular, we show that the Hausdorff dimension of the sample paths of the solution is $\min\{d, \frac{1}{H}\}$ and that the Hausdorff dimension of the level set $L_x = \{t \in [\epsilon, 1] : X_t = x\}$ is $1 - dH$ with positive probability when $dH < 1$.

CONTENTS

1. INTRODUCTION

Random dynamical systems are well established modeling tools for a variety of natural phenomena ranging from physics (fundamental and phenomenological) to chemistry and more recently to biology, economics, engineering sciences and mathematical finance. In many interesting models the lack of any regularity of the external inputs of the differential equation as functions of time is a technical difficulty that hampers their mathematical analysis. The theory of rough paths has been initially developed by T. Lyons [17] in the 1990's to provide a framework to analyze a large class of driven differential equations and the precise relations between the driving signal and the output (that is the state, as function of time, of the controlled system).

Rough paths theory provides a nice framework to study differential equations driven by Gaussian processes (see [9]). In particular, using rough paths theory, we may define solutions of stochastic

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differential equations driven by a fractional Brownian motion. Consider

$$(1.1) \qquad X_t = x + \int_0^t V_0(X_s)ds + \sum_{i=1}^{d} \int_0^t V_i(X_s)dB_s^i,$$

where $x \in \mathbb{R}^n$, $V_0, V_1, \cdots, V_d$ are bounded smooth vector fields on $\mathbb{R}^n$ and $\{B_t, t \geq 0\}$ is a d-dimensional fractional Brownian motion with Hurst parameter $H \in (\frac{1}{4}, 1)$. Existence and uniqueness of solutions to the above equation can be found, for example, in [18]. In particular, when $H = \frac{1}{2}$, this notion of solution coincides with the solution of the corresponding Stratonovitch stochastic differential equation. It is also clear now (cf. [1, 3, 4, 12]) that under Hörmander's condition the law of the solution X_t has a smooth density $p_t(x, y)$ with respect to the Lebesgue measure on $\mathbb{R}^n$.

In this paper, we study fractal properties of the sample paths of solution X to equation (1.1). More specifically, we investigate the Hausdorff dimensions of sample paths $X([0, 1])$ and level sets of X on $[\epsilon, 1]$ for some arbitrarily fixed $\epsilon > 0$ defined as follows:

$$(1.2) \qquad L_x := \{t \in [\epsilon, 1] : X_t = x\}.$$

Our main result extends the classical result for fractional Brownian motions (see e.g. [21] and [22]) and is summarized as follows.

Assumption 1.1. *There exists a strictly positive constant λ such that*

$$v^* V(x) V^*(x) v \geq \lambda |v|^2, \qquad for\ all\ v, x \in \mathbb{R}^n,$$

where we have set $V = (V_j^i)_{i=1,\ldots,n;j=1,\ldots d}$. In particular, under this assumption, $n = d$ in equation (1.1) and $V_1, ..., V_d$ are vector fields on $\mathbb{R}^d$.

Theorem 1.2. *Let X be the solution to equation (1.1). We have almost surely*

$$\dim_{\mathcal{H}} X([0, 1]) = \min\left\{d, \frac{1}{H}\right\}.$$

Moreover, for any given $x \in \mathbb{R}^d$,

  i. *If $dH > 1$, $L_x = \varnothing$ a.s.;*
  ii. *If $dH < 1$, $\dim_{\mathcal{H}} L_x = 1 - dH$ with positive probability.*

Remark 1.3. It is an open problem whether points are polar for X when $dH = 1$. Recently, Dalang, Mueller and Xiao [6] proved the polarity of points in the critical dimension for a family of Gaussian random fields, including fractional Brownian motions. The difficulty in our current setting is that X is neither Makovian nor Gaussian. We believe that X does not hit a given point in the critical dimension (hence $L_x = \varnothing$ a.s.), and it is part of our future work to prove this conjecture.

Let us mention that some relevant results regarding sample path properties of X have been studied in [2] under the same uniform ellipticity condition. In the aforementioned paper the authors have shown that, roughly speaking, there exists $\alpha > 0$ such that for any bounded Borel set $A \subset \mathbb{R}^d$

$$\mathbb{P}\left(X_t \text{ hits } A \text{ for } t \in [a, b]\right) > 0 \quad \Longleftrightarrow \quad \text{Cap}_\alpha(A) > 0,$$

where $\text{Cap}_\alpha(A)$ is the α-dimensional Newtonian capacity of A. In particular, this characterization of hitting probability in terms of capacity implies that $\frac{1}{H}$ is the critical dimension concerning whether or not the process X hits a given point x in $\mathbb{R}^d$; that is, X does not hit x almost surely if

$d > \frac{1}{H}$; and X hits x with positive probability if $d < \frac{1}{H}$. The main ingredient in obtaining these results is upper and lower bound estimates for densities of X.

Based on techniques developed in [2], we are able to have some further study for density functions of X. In particular, we slightly improve the density estimate for the random vector $(X_s, X_t - X_s)$ to an exponential decay (see Theorem 3.4 below)

$$\hat{p}_{s,t-s}(\xi_1, \xi_2) \leq \frac{C}{(t-s)^{dH}} \exp\left\{-\frac{|\xi_2|^{2\gamma}}{C|t-s|^{2\gamma^2}}\right\},$$

for $\gamma < H$. Let us point out that we will prove in Theorem 3.3 below that the density $\tilde{p}_{s,t}(z)$ of $X_t - X_s$ has a faster decay rate in z (We use $\tilde{p}_{s,t}$ because $p_{s,t}$ will be reserved for the joint density of (X_s, X_t) later in this paper.):

$$\tilde{p}_{s,t}(z) \leq \frac{C}{(t-s)^{dH}} \exp\left\{-\lambda\frac{|z|^{(2H+1)\wedge 2}}{C(t-s)^{2H}}\right\}.$$

This is due to the fact that we need to perform integration by parts twice in order to obtain the density of $(X_s, X_t - X_s)$. For this purpose we have to sacrifice the decay rate in order to have an extra order of smoothness in the Malliavin sense.

The organization of the paper is as follows. In Section 2, we give necessary preliminaries on rough paths and Malliavin calculus. In Section 3, we show several tail and density estimates for X that will be needed later in order to obtain our main result. Section 4 is then devoted to the proof of our main result.

2. PRELIMINARY MATEIRAL

For some fixed $H > \frac{1}{4}$, we consider $(\Omega, \mathcal{F}, \mathbb{P})$ the canonical probability space associated with the fractional Brownian motion (in short fBm) with Hurst parameter H. That is, $\Omega = \mathcal{C}_0([0,1])$ is the Banach space of continuous functions vanishing at zero equipped with the supremum norm, $\mathcal{F}$ is the Borel sigma-algebra and $\mathbb{P}$ is the unique probability measure on Ω such that the canonical process $B = \{B_t = (B_t^1, \ldots, B_t^d), \ t \in [0,1]\}$ is a fractional Brownian motion with Hurst parameter H. In this context, let us recall that B is a d-dimensional centered Gaussian process, whose covariance structure is induced by

$$(2.1) \qquad R(t,s) := \mathbb{E}B_s^j B_t^j = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right), \quad s,t \in [0,1] \text{ and } j = 1, \ldots, d.$$

In particular it can be shown, by a standard application of Kolmogorov's criterion, that B admits a continuous version whose paths are γ-Hölder continuous for any $\gamma < H$.

2.1. **Rough paths above fractional Brownian motions.** In this section, we recall some basic results in rough paths theory and how a fractional Brownian motion is lifted to be a rough path. More details can be found in [10] and [18]. For $N \in \mathbb{N}$, recall that the truncated algebra $T^N(\mathbb{R}^d)$ is defined by

$$T^N(\mathbb{R}^d) = \bigoplus_{m=0}^{N} (\mathbb{R}^d)^{\otimes m},$$

with the convention $(\mathbb{R}^d)^{\otimes 0} = \mathbb{R}$. The set $T^N(\mathbb{R}^d)$ is equipped with a straightforward vector space structure plus an multiplication $\otimes$. Let π_m be the projection on the m-th tensor level. Then $(T^N(\mathbb{R}^d), +, \otimes)$ is an associative algebra with unit element $\mathbf{1} \in (\mathbb{R}^d)^{\otimes 0}$.

For $s < t$ and $m \geq 2$, consider the simplex $\Delta_{st}^m = \{(u_1, \ldots, u_m) \in [s, t]^m; u_1 < \cdots < u_m\}$, while the simplices over $[0, 1]$ will be denoted by Δ^m. A continuous map $\mathbf{x} : \Delta^2 \to T^N(\mathbb{R}^d)$ is called a multiplicative functional if for $s < u < t$ one has $\mathbf{x}_{s,t} = \mathbf{x}_{s,u} \otimes \mathbf{x}_{u,t}$. An important example arises from considering paths x with finite variation: for $0 < s < t$ we set

$$\mathbf{x}_{s,t}^m = \sum_{1 \leq i_1, \ldots, i_m \leq d} \left(\int_{\Delta_{st}^m} dx^{i_1} \cdots dx^{i_m} \right) e_{i_1} \otimes \cdots \otimes e_{i_m},$$

where $\{e_1, \ldots, e_d\}$ denotes the canonical basis of $\mathbb{R}^d$, and then define the truncated signature of x as

$$S_N(x) : \Delta^2 \to T^N(\mathbb{R}^d), \qquad (s, t) \mapsto S_N(x)_{s,t} := 1 + \sum_{m=1}^N \mathbf{x}_{s,t}^m.$$

The function $S_N(x)$ for a smooth function x will be our typical example of multiplicative functional. Let us stress the fact that those elements take values in the strict subset $G^N(\mathbb{R}^d) \subset T^N(\mathbb{R}^d)$, called free nilpotent group of step N, and is equipped with the classical Carnot-Caratheodory norm which we simply denote by $|\cdot|$. For a path $\mathbf{x} \in \mathcal{C}([0, 1], G^N(\mathbb{R}^d))$, the p-variation norm of $\mathbf{x}$ is defined to be

$$\|\mathbf{x}\|_{p-\text{var};[0,1]} = \sup_{\Pi \subset [0,1]} \left(\sum_i |\mathbf{x}_{t_i}^{-1} \otimes \mathbf{x}_{t_{i+1}}|^p \right)^{1/p}$$

where the supremum is taken over all subdivisions Π of $[0, 1]$.

With these notions in hand, let us briefly define what we mean by geometric rough path (we refer to [10, 18] for a complete overview): for $p \geq 1$, an element $x : [0, 1] \to G^{\lfloor p \rfloor}(\mathbb{R}^d)$ is said to be a geometric rough path if it is the p-var limit of a sequence $S_{\lfloor p \rfloor}(x^m)$. In particular, it is an element of the space

$$\mathcal{C}^{p-\text{var};[0,1]}([0, 1], G^{\lfloor p \rfloor}(\mathbb{R}^d)) = \{\mathbf{x} \in \mathcal{C}([0, 1], G^{\lfloor p \rfloor}(\mathbb{R}^d)) : \|\mathbf{x}\|_{p-\text{var};[0,1]} < \infty\}.$$

The existence of a geometric rough path over a fractional Brownian motion is proved in [5].

Proposition 2.1. *Let B be a fractional Brownian motion with Hurst parameter $H > \frac{1}{4}$. It admits a lift $\mathbf{B}$ as a geometric rough path of order p for any $p > \frac{1}{H}$.*

2.2. **Malliavin calculus for fractional Brownian motions.** We introduce the basic framework of Malliavin calculus in this subsection. The reader is invited to read the corresponding chapters in [19] for further details. Let $\mathcal{E}$ be the space of $\mathbb{R}^d$-valued step functions on $[0, 1]$, and $\mathcal{H}$ the closure of $\mathcal{E}$ for the scalar product:

$$\langle (\mathbf{1}_{[0,t_1]}, \cdots, \mathbf{1}_{[0,t_d]}), (\mathbf{1}_{[0,s_1]}, \cdots, \mathbf{1}_{[0,s_d]}) \rangle_{\mathcal{H}} = \sum_{i=1}^d R(t_i, s_i).$$

Let $e_1, \ldots, e_d$ be the canonical basis of $\mathbb{R}^d$, there is an isometry $K_H^* : \mathcal{H} \to L^2([0, 1])$ such that

$$K_H^*(\mathbf{1}_{[0,t]} e_i) = \mathbf{1}_{[0,t]} K_H(t, \cdot) e_i,$$

where the kernel K_H is given by

$$K_H(t, s) = c_H \, s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} \, du, \qquad\qquad\qquad H > \frac{1}{2};$$

$$K_H(t, s) = c_{H,1} \left(\frac{s}{t}\right)^{1/2-H} (t-s)^{H-1/2} + c_{H,2} \, s^{1/2-H} \int_s^t (u-s)^{H-\frac{1}{2}} u^{H-\frac{3}{2}} \, du, \quad H \le \frac{1}{2},$$

for some constants c_H, $c_{H,1}$, and $c_{H,2}$.

Let us remark that $\mathcal{H}$ is the reproducing kernel Hilbert space for B. Denote by $\mathscr{H}$ the Cameron-Martin space of B, one can show that the operator $\mathcal{R} := \mathcal{R}_H : \mathcal{H} \to \mathscr{H}$ given by

$$\mathcal{R}\psi := \int_0^{\cdot} K_H(\cdot, s)[K_H^* \psi](s) \, ds$$

defines an isometry between $\mathcal{H}$ and $\mathscr{H}$.

Some isometry arguments allow to define the Wiener integral $B(h) = \int_0^1 \langle h_s, dB_s \rangle$ for any element $h \in \mathcal{H}$, with the additional property $\mathbb{E}[B(h_1)B(h_2)] = \langle h_1, h_2 \rangle_{\mathcal{H}}$ for any $h_1, h_2 \in \mathcal{H}$. A $\mathcal{F}$-measurable real valued random variable F is said to be cylindrical if it can be written, for a given $n \ge 1$, as

$$F = f\left(B(\phi^1), \dots, B(\phi^n)\right),$$

where $\phi^i \in \mathcal{H}$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a C^∞ bounded function with bounded derivatives. The set of cylindrical random variables is denoted $\mathcal{S}$.

The Malliavin derivative is defined as follows: for $F \in \mathcal{S}$, the derivative of F is the $\mathbb{R}^d$ valued stochastic process $(\mathbf{D}_t F)_{0 \le t \le 1}$ given by

$$\mathbf{D}_t F = \sum_{i=1}^n \phi^i(t) \frac{\partial f}{\partial x_i}\left(B(\phi^1), \dots, B(\phi^n)\right).$$

More generally, we can introduce iterated derivatives. If $F \in \mathcal{S}$, we set

$$\mathbf{D}_{t_1, \dots, t_k}^k F = \mathbf{D}_{t_1} \dots \mathbf{D}_{t_k} F.$$

For any $p \ge 1$, it can be checked that the operator $\mathbf{D}^k$ is closable from $\mathcal{S}$ into $\mathbf{L}^p(\Omega; \mathcal{H}^{\otimes k})$. We denote by $\mathbb{D}^{k,p}$ the closure of the class of cylindrical random variables with respect to the norm

$$\|F\|_{k,p} = \left(\mathbb{E}\left(F^p\right) + \sum_{j=1}^k \mathbb{E}\left(\|\mathbf{D}^j F\|_{\mathcal{H}^{\otimes j}}^p\right)\right)^{\frac{1}{p}},$$

and

$$\mathbb{D}^\infty = \bigcap_{p \ge 1} \bigcap_{k \ge 1} \mathbb{D}^{k,p}.$$

Definition 2.2. *Let $F = (F^1, \dots, F^n)$ be a random vector whose components are in $\mathbb{D}^\infty$. Define the Malliavin matrix of F by*

$$\gamma_F = (\langle \mathbf{D}F^i, \mathbf{D}F^j \rangle_{\mathcal{H}})_{1 \le i, j \le n}.$$

Then F is called non-degenerate if γ_F is invertible a.s. and

$$(\det \gamma_F)^{-1} \in \cap_{p \ge 1} L^p(\Omega).$$

It is a classical result that the law of a non-degenerate random vector $F = (F^1, \ldots, F^n)$ admits a smooth density with respect to the Lebesgue measure on $\mathbb{R}^n$. Furthermore, let $C_p^\infty(\mathbb{R}^n)$ be the space of smooth functions whose derivatives together with itself have polynomial growth. The following integration by parts formula allows to get more quantitative estimates (see [19, Proposition 2.1.4]):

Proposition 2.3. *Let $F = (F^1, ..., F^n)$ be a non-degenerate random vector whose components are in $\mathbb{D}^\infty$, and γ_F the Malliavin matrix of F. Let $G \in \mathbb{D}^\infty$ and φ be a function in the space $C_p^\infty(\mathbb{R}^n)$. Then for any multi-index $\alpha \in \{1, 2, ..., n\}^k, k \geq 1$, there exists an element $H_\alpha \in \mathbb{D}^\infty$ such that*

$$\mathbb{E}[\partial_\alpha \varphi(F) G] = \mathbb{E}[\varphi(F) H_\alpha].$$

Here, the elements H_α are recursively given by

$$H_{(i)} = \sum_{j=1}^d \delta\left(G(\gamma_F^{-1})^{ij} \mathbf{D} F^j\right), \quad \text{and} \quad H_\alpha = H_{(\alpha_k)}(H_{(\alpha_1, ..., \alpha_{k-1})}).$$

Here δ is the divergence operator (adjoint of the operator $\mathbf{D}$). Moreover, for $1 \leq p < q < \infty$ we have

$$\|H_\alpha\|_{L^p} \leq C_{p,q} \|\gamma_F^{-1} \mathbf{D} F\|_{k, 2^{k-1}r}^k \|G\|_{k,q},$$

where $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$.

Consider process $W = \{W_t, t \in [0,1]\}$ defined by

$$W_t = B((K_H^*)^{-1}(\mathbf{1}_{[0,t]})).$$

One can show that W is a Wiener process, and the process B has the integral representation

$$B_t = \int_0^t K_H(t,s) dW_s.$$

Based on the above representation, one can consider fractional Brownian motions and hence functionals of fractional Brownian motions as functionals of the underlying Wiener process W. This observation allows us to perform Malliavin calculus with respect to the Wiener process W. We shall perform Malliavin calculus with respect to both B and W. In order to distinguish them, the Malliavin derivatives (and corresponding Sobolev spaces, respectively) with respect to W will be denoted by D (and by $D^{k,p}$, respectively). The relation between the two operators $\mathbf{D}$ and D is given by the following (see e.g. [19, Proposition 5.2.1]).

Proposition 2.4. *Let $D^{1,2}$ be the Malliavin-Sobolev space corresponding to the Wiener process W. Then $\mathbb{D}^{1,2} = D^{1,2}$, and for any $F \in D^{1,2}$ we have $\mathbf{D}F = K^* \mathbf{D}F$ whenever both members of the relation are well defined.*

In order to estimate the bivariate density function for the random vector (X_s, X_t), we will need a version of conditional integration by parts formula. For this purpose, we choose to work with the underlying Wiener process W. The advantage of doing so is that projections on subspaces are easier to describe in a L^2 type setting.

It is known that B and the Wiener process W generate the same filtration which we denote by $\{\mathcal{F}_t; t \in [0,1]\}$. Set $L_t^2 \equiv L^2([t,1])$ and $\mathbb{E}_t = \mathbb{E}(\cdot | \mathcal{F}_t)$. For a random variable F and $t \in [0,1]$,

define, for $k \geq 0$ and $p > 0$, the conditional Sobolev norm

$$\|F\|_{k,p;t} = \left(\mathbb{E}_t\left[F^p\right] + \sum_{j=1}^{k} \mathbb{E}_t\left[\left\| D^j F \right\|^p_{(L_t^2)^{\otimes j}} \right] \right)^{\frac{1}{p}}.$$

By convention, when $k = 0$ we always write $\|F\|_{p;t} = \|F\|_{0,p;t}$. The conditional Malliavin matrix of F is given by

$$(2.2) \qquad \Gamma_{F,t} = \left(\langle DF^i, DF^j \rangle_{L_t^2} \right)_{1 \leq i, j \leq n}.$$

The following proposition is a conditional version of Proposition 2.3.

Proposition 2.5. *Fix $k \geq 1$. Let $F = (F^1, ..., F^n)$ be a random vector and G a random variable. Assume both F and G are smooth in the Malliavin sense and $(\det \Gamma_{F,s})^{-1}$ has finite moments of all orders. Then for any multi-index $\alpha \in \{1, \ldots, n\}^k, k \geq 1$, there exists an element $H_\alpha^s(F, G) \in \cap_{p \geq 1} \cap_{m \geq 0} D^{m,p}$ such that*

$$\mathbb{E}_s\left[(\partial_\alpha \varphi)(F)\, G \right] = \mathbb{E}_s\left[\varphi(F)\, H_\alpha^s(F, G) \right], \qquad \varphi \in \mathcal{C}_p^\infty(\mathbb{R}^d),$$

where $H_\alpha^s(F, G)$ is recursively defined by

$$H_{(i)}^s(F, G) = \sum_{j=1}^{n} \delta_s \left(G \left(\Gamma_{F,s}^{-1} \right)_{ij} DF^j \right), \quad H_\alpha^s(F, G) = H_{(\alpha_k)}^s(F, H_{(\alpha_1, ..., \alpha_{k-1})}^s(F, G)).$$

Here δ_s denotes the Skorohod integral with respect to the Wiener process W on the interval $[s, 1]$. (See [19, Section 1.3.2] for a detailed account of the definition of δ_s.) Furthermore, the following norm estimate holds true:

$$\|H_\alpha^s(F, G)\|_{p;s} \leq c_{p,q} \|\Gamma_{F,s}^{-1} DF\|_{k,2^{k-1}r;s}^k \|G\|_{k,q;s}^k,$$

where $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$.

3. TAIL AND DENSITY ESTIMATES

Consider the following stochastic differential equation driven by a fractional Brownian motion with Hurst parameter $H > \frac{1}{4}$,

$$(3.1) \qquad X_t = x + \sum_{i=1}^{d} \int_0^t V_i(X_s) dB_s^i + \int_0^t V_0(X_s) ds.$$

Here $V_i, i = 0, 1, ..., d$ are C^∞-bounded vector fields on $\mathbb{R}^d$ which form a uniform elliptic system (recall Assumption 1.1). Proposition 2.1 ensures the existence of a lift of B as a geometrical rough path, which allows one to consider equation (3.1) as a rough differential equation. Then general rough paths theory (see, e.g., [10, 11]) together with some integrability results ([4, 8]) shows that equation (3.1) admits a unique finite p-var continuous solution X in the rough paths sense, for any $p > \frac{1}{H}$. Moreover, there exists $\lambda > 0$ such that

$$\mathbb{E}\left[\exp \lambda \left(\sup_{t \in [0,1]} |X_t|^{(2H+1)\wedge 2} \right) \right] < \infty.$$

The above moment estimate (or rather tail estimate) extends to the increments of X.

Proposition 3.1. *There exist nonnegative constants C and c not depending on $s, t \in [0, 1]$ and ξ such that*

$$\mathbb{P}\left(\sup_{s \leq u < v \leq t} |X_v - X_u| \geq \xi\right) \leq C \exp\left\{-\frac{c\xi^{(2H+1)\wedge 2}}{(t-s)^{2H}}\right\}.$$

Proof. The claimed result was proved in [8] for the case when there is no drift. Below we show that their argument can be modified to include the drift. First consider the case $\frac{1}{4} < H < \frac{1}{2}$. Fix $p > \frac{1}{H}$ and define the control

$$\omega(u, v) = \|\mathbf{B}\|^p_{p-\text{var};[u,v]} + (v - u).$$

By Theorem 12.6 in [10], there exists constant c_V only depends on the vector fields $V_i's$ such that

$$(3.2) \qquad \|X\|_{p-\text{var};[u,v]} \leq c_V \left(\omega(u, v)^{1/p} \vee \omega(u, v)\right).$$

Fix $\alpha \geq 1$ and construct a partition of $[s, t]$ inductively in the following way: we set $t_0 = s$ and

$$t_{i+1} = \inf\left\{u > t_i; \|\mathbf{B}\|^p_{p-\text{var};[t_i,u]} \geq \alpha\right\}.$$

Let $N_{\alpha,s,t,p} = \min\{n \geq 0; t_n \geq t\}$ and $X_{t_i,t_{i+1}} = X_{t_{i+1}} - X_{t_i}$. Since we have taken $\alpha \geq 1$ and $t_{i+1} - t_i \leq 1$, inequality (3.2) implies

$$|X_{t_i,t_{i+1}}| \leq c_V \,\omega(t_i, t_{i+1}) \leq c_V(\alpha + 1).$$

Now for any fixed $u < v$ in $[s, t]$, suppose l and m are such that $u \in [t_l, t_{l+1}]$ and $v \in [t_m, t_{m+1}]$. We have

$$|X_u - X_v| \leq \sum_{i=0}^{l} |X_{t_i,t_{i+1}}| + |X_{t_l,u}| + |X_{u,t_{l+1}}|$$

$$+ \sum_{i=l+1}^{m} |X_{t_i,t_{i+1}}| + |X_{t_m,v}| + |X_{v,t_{m+1}}| + \sum_{i=m+1}^{N_{\alpha,s,t,p}} |X_{t_i,t_{i+1}}| \leq c_V(\alpha + 1)(N_{\alpha,t,p} + 2).$$

Since fractional Brownian motions have stationary increments, by Corollary 2 in [8] (or [4, Theorem 6.6]) there exists constants C and c not depending on s, t and ξ such that

$$(3.3) \qquad \mathbb{P}\left(N_{\alpha,s,t,p} + 2 > \xi\right) \leq C \exp\left\{-\frac{c\xi^{2H+1}}{(t-s)^{2H}}\right\}.$$

This easily yields

$$\mathbb{P}\left(\sup_{s \leq u < v \leq t} |X_v - X_u| \geq \xi\right) \leq \mathbb{P}\left(c_V(\alpha + 1)(N_{\alpha,t,p} + 2) > \xi\right) \leq C \exp\left\{-\frac{c\xi^{2H+1}}{(t-s)^{2H}}\right\},$$

which is our claim. The case $H > \frac{1}{2}$ is handled along the same lines, except that the coefficient ξ^{2H+1} in (3.3) is replaced by ξ^2. $\qquad \square$

The vector X_t is a typical example of a smooth random variable in the Malliavin sense. Recall that D is the Malliavin derivative operator with respect to the underlying Wiener process W and $\Gamma_{F,t}$ is defined in (2.2) for a random variable F. The following estimate is a restatement of Proposition 5.9 in [2].

Lemma 3.2. *Let $\epsilon \in (0,1)$, and consider $H \in (1/4,1)$. Then there exist constants $C, r > 0$ depending on ϵ such that for $\epsilon \leq s \leq t \leq 1$ the following holds:*

$$\|\Gamma^{-1}_{X_t - X_s, s}\|^d_{d, 2^{d+2}; s} \leq \frac{C}{(t-s)^{2dH}} \left(\mathbb{E}_s(1+G)\right)^{\frac{d}{2^{d+2}}}$$

$$\|D(X_t - X_s)\|^d_{d, 2^{d+2}; s} \leq C(t-s)^{dH} \left(\mathbb{E}_s(1+G)\right)^{\frac{d}{2^{d+2}}},$$

where G is a random variable smooth in the Malliavin sense and has finite moments to any order.

With the above lemma in hand, we are able to obtain an upper bound for the density of $X_t - X_s$.

Theorem 3.3. *Fix $\epsilon \in (0,1)$. Let $\tilde{p}_{s,t}(z)$ be the density function of $X_t - X_s$. There exits a positive constant C depending only on ϵ such that*

$$\tilde{p}_{s,t}(z) \leq \frac{C}{(t-s)^{dH}} \exp\left\{ -\frac{|z|^{(2H+1)\wedge 2}}{C(t-s)^{2H}} \right\}, \quad \text{for all } \epsilon \leq s < t \leq 1.$$

Proof. We first write

$$\tilde{p}_{s,t}(z) = \mathbb{E}\left[\delta_z(X_t - X_s)\right]$$
$$= \mathbb{E}\mathbb{E}_s\left[\delta_z(X_t - X_s)\right].$$

Next, we bound $M_{st} = \mathbb{E}_s\left[\delta_z(X_t - X_s)\right]$ by first using the conditional integration by parts formula in Proposition 2.5 for the Dirac delta function $\varphi = \delta_z$ and $F = X_t - X_s, G = 1$, and then Cauchy-Schwarz inequality:

$$|M_{s,t}| = |\mathbb{E}_s[\mathbf{1}_{(X_t - X_s > \xi_2)} H^s_{(1,\dots,d)}(X_t - X_s, 1)]|$$
$$\leq C\|\Gamma^{-1}_{X_t - X_s, s}\|^d_{d, 2^{d+2}; s} \|D(X_t - X_s)\|^d_{d, 2^{d+2}; s} \left(\mathbb{E}_s[\mathbf{1}_{(X_t - X_s > \xi_2)}]\right)^{1/2}.$$

Note that though it requires $\varphi \in \mathcal{C}^\infty_p(\mathbb{R}^d)$ in Proposition 2.5, it could pass to $\varphi = \delta_z$ by a suitable approximation argument. Thus, owing to Lemma 3.2 we obtain:

$$(3.4) \qquad \tilde{p}_{s,t}(z) \leq \frac{C}{(t-s)^{dH}} \mathbb{E}\left[\left(\mathbb{E}_s(1+G)^2\right)^{\frac{d}{2^{d+1}}} \left(\mathbb{E}_s[\mathbf{1}_{(X_t - X_s > \xi_2)}]\right)^{1/2} \right].$$

Then an application of Cauchy-Schwarz inequality together with Proposition 3.1 finishes the proof. $\square$

Finally we slightly improve the estimate in [2] for the bivariate density of (X_s, X_t). The improvement is made possible by considering the Besov norms of $\mathbf{B}$, instead of the process $\mathbf{M}$ in [2], to obtain a better concentration inequality for $X_t - X_s$.

Theorem 3.4. *Fix $\epsilon \in (0,1)$ and $\gamma < H$. Let $p_{s,t}(z_1, z_2)$ be the joint density of the random vector (X_s, X_t) for $\epsilon \leq s < t \leq 1$. There is a positive constant C depending only on ϵ such that*

$$p_{s,t}(z_1, z_2) \leq \frac{C}{(t-s)^{dH}} \exp\left\{ -\frac{|z_1 - z_2|^{2\gamma}}{C|t-s|^{2\gamma^2}} \right\}, \quad \text{for all } \epsilon \leq s < t \leq 1.$$

Proof. The proof is similar to the one in the above, but with a slightly more careful estimate for the tail probability of $X_t - X_s$.

First note the existence and smoothness of the density function $p_{s,t}(z_1, z_2)$ a consequence of Lemma 3.2. We then write

$$p_{s,t}(z_1, z_2) = \hat{p}_{s,t-s}(z_1, z_2 - z_1), \quad \text{for } z_1, z_2 \in \mathbb{R}^d,$$

where $\hat{p}_{s,t-s}(\cdot, \cdot)$ denotes the joint density of the random vector $(X_s, X_t - X_s)$. We now bound the function $\hat{p}_{s,t-s}$, which shall be expressed as

$$\hat{p}_{s,t-s}(\xi_1, \xi_2) = \mathbb{E}\left[\delta_{\xi_1}(X_s)\, \delta_{\xi_2}(X_t - X_s)\right], \quad \text{for } \xi_1, \xi_2 \in \mathbb{R}^d,$$
$$= \mathbb{E}\left[\delta_{\xi_1}(X_s)\, \mathbb{E}_s\left[\delta_{\xi_2}(X_t - X_s)\right]\right].$$

As before, we can bound $M_{st} = \mathbb{E}_s\left[\delta_{\xi_2}(X_t - X_s)\right]$ as follows.

$$|M_{s,t}| \le C\|\Gamma^{-1}_{X_t - X_s, s}\|^d_{d, 2^{d+2}, s}\, \|D(X_t - X_s)\|^d_{d, 2^{d+2}; s}\left(\mathbb{E}_s[\mathbf{1}_{(X_t - X_s > \xi_2)}]\right)^{1/2}.$$

Thus, owing to Lemma 3.2 we obtain:

$$(3.5) \qquad \hat{p}_{s,t-s}(\xi_1, \xi_2) \le \frac{C}{(t-s)^{nH}}\mathbb{E}\left[\delta_{\xi_1}(X_s)\left(\mathbb{E}_s(1+G)^2\right)^{\frac{d}{2^{d+1}}}\left(\mathbb{E}_s[\mathbf{1}_{(X_t - X_s > \xi_2)}]\right)^{1/2}\right]$$

To proceed, recall that $\mathbf{B}$ is the lift of B as a rough path. Set

$$\mathcal{N}_{\gamma, q}(\mathbf{B}) = \int_0^1 \int_0^1 \frac{d(\mathbf{B}_v, \mathbf{B}_u)^q}{|v - u|^{\gamma q}}\, du dv,$$

where $\gamma < H$ and q a nonnegative integer. The reason we need to consider these Besov norms $\mathcal{N}_{\gamma, q}(\mathbf{B})$ is that they are smooth in the Malliavin sense. Indeed, denote by $[1/H]$ the integer part of $1/H$, the Besov norm $\mathcal{N}_{\gamma, 2q}(\mathbf{B})$ is in the inhomogeneous Wiener chaos of order $[1/H]2q$ for all q, hence a smooth element in the Malliavin sense (see, e.g., the discussion in [13, Section 5]).

Denote by $\|\mathbf{B}\|_{\gamma; [0,1]}$ the Hölder norm of order γ (of $\mathbf{B}$) on the unit interval. By the Garsia-Rodemich-Rumsey inequality in Carnot groups ([10, Proposition A. 8]) we have for any $\epsilon > 0$ such that $\gamma + \epsilon < H$, and $q > 1/\epsilon$,

$$\|\mathbf{B}\|_{\gamma; [0,1]} \le C\mathcal{N}_{\gamma+\epsilon, 2q}(\mathbf{B})^{1/2q}.$$

Hence, it is readily checked by (3.2) that (note that $0 < t - s < 1$ in our case)

$$|X_t - X_s| \le C\, |t - s|^\gamma(1 + \mathcal{N}_{\gamma+\epsilon, 2q}(\mathbf{B})^{1/2q})^{1/\gamma}.$$

Furthermore, by [7], for any $\gamma < H$ there exists a constant $\lambda_0 > 0$ such that

$$\mathbb{E}\exp\left\{\lambda_0 \|\mathbf{B}\|^2_{\gamma; [0,1]}\right\} < \infty.$$

This together with

$$\mathcal{N}_{\gamma+\epsilon, 2q}(\mathbf{B})^{1/2q} \le \|\mathbf{B}\|_{\gamma+\epsilon; [0,1]},$$

implies that $\mathcal{N}_{\gamma+\epsilon, 2q}(\mathbf{B})^{1/2q}$ has Gaussian tail. Thus, for $\lambda < \lambda_0$ we can find a constant $C > 0$,

$$\mathbb{E}_s\left[\mathbf{1}_{(X_t - X_s > \xi_2)}\right] \le C\exp\left\{-\lambda\frac{|\xi|^{2\gamma}}{C|t - s|^{2\gamma^2}}\right\}\mathbb{E}_s\exp\left\{\lambda(1 + \mathcal{N}_{\gamma+\epsilon, 2q}(\mathbf{B}))^{2/2q}\right\}.$$

Plugging this inequality into (3.5), we end up with:

$$(3.6) \qquad \hat{p}_{s,t-s}(\xi_1, \xi_2) \le \frac{C}{(t-s)^{dH}}\exp\left\{-\lambda\frac{|\xi|^{2\gamma}}{C|t - s|^{2\gamma^2}}\right\}\mathbb{E}\left[\delta_{\xi_1}(X_s)\, \Psi_1 \Psi_2\right]$$

where Ψ_1 and Ψ_2 are two random variables which are smooth in the Malliavin calculus sense. Also note that we may need to choose λ even smaller to make sure Malliavin derivatives of Φ_2 have moments to certain large order. Based on the above consideration, we can now integrate (3.6) safely by parts in order to regularize the term $\delta_{\xi_1}(X_s)$, which finishes the proof.

$\square$

To close the discussion in this section, let us state a mild lower bound (strict positivity) of the density of X_t. We direct the interested reader to [2, Theorem 1.4] for its proof.

Theorem 3.5. *Consider the solution X to equation* (3.1) *driven by a d-dimensional fractional Brownian motion with Hurst parameter $H > \frac{1}{4}$. For each fixed $x \in \mathbb{R}^d$, denote by $p_t(x, \cdot) : \mathbb{R}^d \to \mathbb{R}_+$ the density function of the random variable X_t. Assume that Assumption 1.1 is satisfied. Then $p_t(x, y) > 0$ for all $y \in \mathbb{R}^d$.*

4. FRACTAL DIMENSIONS OF SDES DRIVEN BY FRACTIONAL BROWNIAN MOTIONS

We first briefly recall the definition of capacity and packing dimension, as well as their connection to Hausdorff dimension. A kernel κ is a measurable function $\kappa : \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty]$. For a Borel measure μ on $\mathbb{R}^d$, the energy of μ with respect to the kernel κ is defined by

$$\mathcal{E}_\kappa(\mu) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \kappa(x, y) \mu(dx) \mu(dy).$$

For any Borel set $E \subset \mathbb{R}^d$, the capacity of E with respect to κ, denoted by $\mathcal{C}_\kappa(E)$, is defined by

$$\mathcal{C}_\kappa(E) = \left[\inf_{\mu \in \mathcal{P}(E)} \mathcal{E}_\kappa(\mu) \right]^{-1}.$$

Here $\mathcal{P}(E)$ is the family of probability measures carried by E. Note that $\mathcal{C}_\kappa(E) > 0$ if and only if there is a probability measure μ on E with finite κ-energy. Throughout our discussion, we will mostly consider the case when $\kappa(x, y) = f(|x - y|)$, where

$$f(r) = \begin{cases} r^{-\alpha}, & \text{if } \alpha > 0; \\ \log\left(\frac{e}{r \wedge 1}\right), & \text{if } \alpha = 0. \end{cases}$$

The corresponding energy and capacity will be denoted by $\mathcal{E}_\alpha(\mu)$ and $\mathcal{C}_\alpha(E)$, respectively; and the former will be called the α-energy of μ and the latter will be called the Bessel-Riesz capacity of E of order α. The capacity dimension of E is defined by

$$\dim_c(E) = \sup\{\alpha > 0; \mathcal{C}_\alpha(E) > 0\}.$$

It is know by Frostman's theorem (cf. [14] or [15]) that

$$\dim_{\mathcal{H}} E = \dim_c(E),$$

for every compact subset E of $\mathbb{R}^d$. Hence, in order to show $\dim_{\mathcal{H}} E \geq \alpha$ one only needs to find a finite measure μ on E such that the α-energy of μ is finite.

Packing dimension and packing measure were introduced as dual concept to Hausdorff dimension and Hausdorff measure. Denote by $\dim_{\mathcal{P}}$ by packing dimension. For any $\varepsilon > 0$ and any

bounded set $F \subset \mathbb{R}^d$, let $N(F, \varepsilon)$ be the smallest number of balls of radius ε (in Euclidean metric) needed to cover F. Then the upper box-counting dimension of F is

$$\overline{\dim}_{\mathcal{B}} F = \limsup_{\varepsilon \downarrow 0} \frac{\log N(F, \varepsilon)}{-\log \varepsilon}.$$

The packing dimension of F can be defined by

$$\dim_{\mathcal{P}} F = \inf \left\{ \sup_n \overline{\dim}_{\mathcal{B}} F_n : F \subset \cup_{n=1}^{\infty} F_n \right\}.$$

It is known that for any bounded set $F \subset \mathbb{R}^d$ (cf. [22]),

(4.1) $\dim_{\mathcal{H}} F \le \dim_{\mathcal{P}} F \le \overline{\dim}_{\mathcal{B}} F \le d.$

4.1. **Hausdorff dimension for SDE driven by fBm.** Recall that X is the solution to

(4.2) $$X_t = x + \int_0^t V_0(X_s) ds + \sum_{i=1}^{d} \int_0^t V_i(X_s) dB_s^i,$$

where $V_i, i = 0, 1, ..., d$ are C^∞-bounded vector fiefs on $\mathbb{R}^d$ satisfying Assumption 1.1. Denote by

$$X([0, 1]) = \{X(t) : t \in [0, 1]\}$$

the collection of sample paths of X on time interval $[0, 1]$.

Theorem 4.1. *We have*

$$\overline{\dim}_{\mathcal{B}} X([0, 1]) \le \frac{1}{H}, \quad \text{a.s.}$$

Proof. For any constant $0 < \gamma < H$, we have

(4.3) $$\sup_{s,t \in [0,1]} \frac{|X_t - X_s|}{|s - t|^\gamma} = \|X\|_{\gamma;[0,1]},$$

and $\|X\|_{\gamma;[0,1]}$ has moments to any order. We fix a sample path w and suppress it. For any integer $n \ge 2$, we divide $[0, 1]$ into m_n sub-intervals $\{R_{n,i}\}$ with length $n^{-1/H}$. Here when $n^{1/H}$ is not an integer, we allow the last sub-interval has length less than $n^{-1/H}$. Clearly, we can find $c_1 > 1$ such that

$$m_n \le c_1 n^{\frac{1}{H}},$$

and $X([0, 1])$ can be covered by $X(R_{n,i}), 1 \le i \le m_n$. By (4.3), we see that the diameter of the image $X(R_{n,i})$ is controlled by

$$\text{diam} X(R_{n,i}) \le c_2 n^{-1+\delta},$$

where $\delta = (H - \gamma)/H$. Consequently, for $\varepsilon_n = c_2 n^{-1+\delta}$, $X([0, 1])$ can be covered by at most m_n balls in $\mathbb{R}^d$ of radius ε_n. That is

$$N(X([0, 1], \varepsilon_n)) \le c_1 n^{\frac{1}{H}}.$$

This implies

$$\overline{\dim}_{\mathcal{B}} X([0, 1]) \le \frac{1}{1 - \delta} \frac{1}{H}, \quad \text{a.s.}$$

Letting $\gamma \uparrow H$ finishes the proof.

$\square$

Next, we turn to the lower bound.

Theorem 4.2. *Let X be the solution to equation (4.2). We have*

$$\dim_{\mathcal{H}} X([0,1]) \geq \min\left\{d, \frac{1}{H}\right\}.$$

Proof. Note that for any $\varepsilon > 0$, we have $\dim_{\mathcal{H}} X([0,1]) \geq \dim_{\mathcal{H}} X([\varepsilon,1])$. To prove the claimed result, it suffices to show that

$$\dim_{\mathcal{H}} X([\varepsilon,1]) \geq \gamma \quad \text{a.s.}$$

for every $0 < \gamma < \min\{d, 1/H\}$.

Let μ_X be the image of measure of the Lebesgue measure on $[\varepsilon, 1]$ under the mapping $t \mapsto X_t$. Then the energy of μ_X of order γ can be written as

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\mu_X(dx)\mu_X(dy)}{|x-y|^{\gamma}} = \int_{[\varepsilon,1]} \int_{[\varepsilon,1]} \frac{dsdt}{|X_t - X_s|^{\gamma}}.$$

Hence, by Frostman's theorem, it is sufficient to show that for every $0 < \gamma < \min\{d, 1/H\}$,

$$(4.4) \qquad \mathcal{E}_{\gamma} = \int_{[\varepsilon,1]} \int_{[\varepsilon,1]} \mathbb{E}\left(\frac{1}{|X_t - X_s|^{\gamma}}\right) dsdt < \infty.$$

By Theorem 3.3,

$$\mathbb{E}\frac{1}{|X_t - X_s|^{\gamma}} \leq \int_{\mathbb{R}^d} \frac{1}{|z|^{\gamma}} \frac{C}{|t-s|^{dH}} e^{-\frac{|z|^{(2H+1)\wedge 2}}{C|t-s|^{2H}}} dz$$

$$= \frac{C}{|t-s|^{\gamma H}} \int_{\mathbb{R}^d} \frac{1}{|\xi|^{\gamma}} \exp\left\{-\frac{|\xi|^{(2H+1)\wedge 2}}{C|t-s|^{H(2-(2H+1)\wedge 2)}}\right\} d\xi \qquad (\xi = z/(t-s)^H)$$

$$\leq \frac{C'}{|t-s|^{\gamma H}}.$$

In the above, we have used the fact that $\gamma < d$ to ensure the integration near 0 is finite. We have also used that $2 - (2H+1) \wedge 2 \geq 0$ to ensure the integration at infinity is finite and uniform in $s, t \in [0,1]$. Now plug the above into (4.4) and note that we assume $\gamma < \frac{1}{H}$. The proof is thus completed. $\qquad \square$

Theorem 4.1 and Theorem 4.2 together with (4.1) give us the Hausdorff dimension of sample path of X.

Theorem 4.3. *Let X be the solution to equation (4.2). We have almost surely*

$$\dim_{\mathcal{H}} X([0,1]) = \min\left\{d, \frac{1}{H}\right\}.$$

It is expected that similar argument also allows us to analyze the Hausdorff dimension of $\mathrm{Gr} X([0,1]) = \{(t, X(t)) : t \in [0,1]\}$, the graph of X on $[0,1]$. The result is summarized in the following theorem.

Theorem 4.4. *The following holds regarding the dimension of the graph of X.*

$$\dim_{\mathcal{H}} \mathrm{Gr} X([0,1]) = \min\left\{(1-H)d + 1, \frac{1}{H}\right\}.$$

i.e.,

$$\dim_{\mathcal{H}} \operatorname{Gr} X([0,1]) = \begin{cases} \frac{1}{H}, & Hd > 1; \\ (1-H)d+1, & Hd < 1 \end{cases}$$

Proof. We first prove the upper bound of $\dim_{\mathcal{H}} \operatorname{Gr}(X[0,1])$. Given any $\delta > 0$, we can cover $\operatorname{Gr} X([0,1])$ by m_n many balls in $\mathbb{R}^{1+d}$ with radius $n^{-1+\delta}$, where $m_n \leq cn^{1/H}$, which provides an upper bound $1/H$ for $\dim \operatorname{Gr} X([0,1])$. To show the other upper bound $(1-H)d+1$, we notice that as in the proof to Theorem 4.1, each $R_{n,i} \times X_{R_{n,i}}$ can be covered by l_n balls with radius $n^{-1/H}$, where

$$l_n = n^{\left(\frac{1}{H}+(-1+\delta)\right)d+1+\frac{1}{H}}.$$

Therefore $\operatorname{Gr} X([0,1])$ can be covered by $m_n \times l_n$ balls with radius $n^{-1/H}$. Letting $\delta > 0$ go to zero proves that

$$\dim_{\mathcal{H}} \operatorname{Gr} X([0,1]) \leq H\left[\left(\frac{1}{H}-1\right)d + \frac{1}{H}\right] = 1 + (1-H)d.$$

Now we need to prove the lower bound for $\dim_{\mathcal{H}} \operatorname{Gr} X([0,1])$. Since $\dim_{\mathcal{H}} \operatorname{Gr} X \geq \dim_{\mathcal{H}} X$ always holds, when $(1-H)d+1 > 1/H$, or equivalently, $Hd > 1$, $\dim_{\mathcal{H}} \operatorname{Gr} X \geq \dim_{\mathcal{H}} X = 1/H$. Therefore we only need to worry about the case when $(1-H)d+1 < 1/H$, or equivalently, $Hd < 1$. For any $\gamma < 1 + (1-H)d$ arbitrarily fixed, we will prove $\dim_{\mathcal{H}} \operatorname{Gr} X([0,1])$ a.s.. By Frostman's theorem, we will show

$$(4.5) \qquad \int_{t=0}^{1} \int_{s=0}^{1} \mathbb{E}\left[\frac{1}{(|t-s|^2 + |X_t - X_s|^2)^{\gamma/2}}\right] ds dt < \infty.$$

By Theorem 3.3, the upper bound density estimate for $X_t - X_s$, we have for the left hand side of (4.5) that for any $\delta > 0$ such that $\gamma + dH < 1 - \delta + d$,

$$\int_{t=0}^{1} \int_{s=0}^{1} \mathbb{E}\left[\frac{1}{(|t-s|^2 + |X_t - X_s|^2)^{\gamma/2}}\right] ds dt$$

$$\leq \int_{t=0}^{1} \int_{s=0}^{1} \int_{X_t - X_s := y \in \mathbb{R}^d} \frac{1}{|t-s|^{dH}} e^{-\frac{|y|^{(2H+1)\wedge 2}}{|t-s|^{2H}}} \left(\frac{1}{(|t-s|^2 + |y|^2)}\right)^{\gamma/2} dy ds dt$$

$$\leq \int_{t=0}^{1} \int_{s=0}^{1} \int_{X_t - X_s := y \in \mathbb{R}^d} \frac{1}{|t-s|^{dH}} e^{-\frac{|y|^{(2H+1)\wedge 2}}{|t-s|^{2H}}} \frac{1}{|t-s|^{\gamma}} dy ds dt$$

$$\leq \int_{t \in [0,1]} \int_{s \in [0,1]} \int_{y \in \mathbb{R}^d} \frac{1}{|t-s|^{dH+\gamma}} e^{-\frac{|y|^{(2H+1)\wedge 2}}{|t-s|^{2H}}} dy ds dt$$

$$\leq \int_{t \in [0,1]} \int_{s \in [0,1]} \int_{y \in \mathbb{R}^d} \frac{1}{|t-s|^{1-\delta+d}} e^{-\frac{|y|^{(2H+1)\wedge 2}}{|t-s|^{2H}}} dy ds dt$$

$$\leq \int_{t \in [0,1]} \int_{s \in [0,1]} \frac{c}{|t-s|^{1-\delta}} \int_{r \in [0,1]} \frac{r^{d-1}}{|t-s|^d} e^{-\frac{r^{(2H+1)\wedge 2}}{|t-s|^{2H}}} dr ds dt \quad \text{(Using polar coordinates)}$$

$$\leq \int_{t \in [0,1]} \int_{s \in [0,1]} \frac{c}{|t-s|^{1-\delta}} \int_{\xi \in [0,1]} e^{-\xi^{2H/d}} d\xi ds dt < \infty \qquad \left(\xi = \frac{r^d}{|t-s|^d}\right).$$

The fact that $2H < (2H + 1) \wedge 2$ was used in the last "$\leq$". Therefore it has been shown that when $(1 - H)d + 1 < 1/H$, i.e, when $Hd < 1$, $\dim_{\mathcal{H}} \mathrm{Gr}X([0, 1]) \geq (1 - H)d + 1$. This proves that

$$\dim_{\mathcal{H}} \mathrm{Gr}X([0, 1]) \geq \min \left\{ (1 - H)d + 1, \frac{1}{H} \right\}.$$

$\square$

Remark 4.5. The same result on Hausdorff dimension of sample path and graph is proved for solution of elliptic differential equations driven by Brownian motion in [16, Theorem 5.1].

4.2. **Hausdorff dimension of level sets.** In this section, we study Hausdorff dimension of the level sets of processes driven by fBM. Before we state the main results, we have the following two lemmas regarding the upper and lower bounds of Hausdorff dimensions of the level sets which actually matches each other. Recall that in (1.2) we have defined level sets $L_x = \{t \in [\epsilon, 1] : X_t = x\}$ of X on $[\epsilon, 1]$, where $0 < \epsilon < 1$ is arbitrarily fixed. The following lemma addresses the upper bound.

Lemma 4.6. *Let X be the solution to (4.2). We have almost surely*

(4.6)
$$\overline{\dim}_{\mathcal{B}} L_x \leq 1 - dH,$$

when $dH < 1$; and $L_x = \varnothing$ when $dH > 1$.

Proof. For a fixed integer $n \geq 1$, we divide the interval $[0, 1]$ into m_n subintervals I_i, $1 \leq i \leq m_n$ with lengths $n^{-1/H}$. $m_n \leq cn^{1/H}$ for some $c > 0$. Let $0 < \delta < 1$ be fixed, and let t_i be the left endpoint of I_i. In the following proof, c is always some strictly positive constant which may change from line to line. We have for $1 \leq i \leq m_n$

(4.7)
$$\mathbb{P}\left(x \in X(I_i)\right) \leq \mathbb{P}\left(\max_{s,t \in I_i} |X_s - X_t| \leq n^{-(1-\delta)}; x \in X(I_i)\right) + \mathbb{P}\left(\max_{s,t \in I_i} |X_s - X_t| > n^{-(1-\delta)}\right).$$

We first claim that the first term on the right hand side of (4.7) is bounded by $cn^{-(1-\delta)}$. Assuming the process is started at z and applying Theorem 1.5 in [2], we have for $1 \leq i \leq m_n$,

$$\mathbb{P}\left(\max_{s,t \in I_i} |X_s - X_t| \leq n^{-(1-\delta)}; x \in X(I_i)\right) \leq \mathbb{P}\left(|X_{t_i} - x| \leq n^{-(1-\delta)}\right)$$

$$\leq \int_{y \in B(x, n^{-(1-\delta)})} \frac{c}{t_i^{dH}} \exp\left(-\frac{|y - z|^{(2H+1)\wedge 2}}{ct_i^{2H}}\right) dy$$

$$\leq \int_{y \in B(x, n^{-(1-\delta)})} \frac{c}{t_i^{dH}} \exp\left(-\frac{|y - x|^{(2H+1)\wedge 2}}{ct_i^{2H}}\right) dy$$

$$(\text{Using polar coordinates}) \leq c \int_{r=0}^{n^{-(1-\delta)}} \frac{r^{d-1}}{t_i^{dH}} \exp\left(-\frac{r^{2\wedge(2H+1)}}{ct_i^{2H}}\right) dr \leq cn^{-(1-\delta)d},$$

where the last inequality is due to the fact that t_i is bounded away from zero by at least ϵ.

For the second term on the right hand side of (4.7), we would like to bound it from above by $e^{-cn^{2\delta}}$, which will then bound the left hand side of (4.7) by $cn^{-(1-\delta)d}$. Indeed, it follows from

Proposition 3.1 that for $1 \le i \le m_n$,

$$\mathbb{P}\left[\max_{s,t\in I_i}|X_s - X_t| > n^{-(1-\delta)}\right] \le c\exp\left(-\frac{cn^{-(1-\delta)[(2H+1)\wedge 2]}}{n^{-\frac{1}{H}(2H)}}\right) \le \exp\left[-c\left(\frac{n^{-2(1-\delta)}}{n^{-2}}\right)\right] = e^{-cn^{2\delta}}.$$

Therefore, we have shown that

$$\mathbb{P}\left(x \in X(I_i)\right) \le \mathbb{P}\left(\max_{s,t\in I_i}|X_s - X_t| \le n^{-(1-\delta)}; x \in X(I_i)\right) + \mathbb{P}\left(\max_{s,t\in I_i}|X_s - X_t| > n^{-(1-\delta)}\right)$$

(4.8) $\le cn^{-(1-\delta)d} + e^{-cn^{2\delta}} \le cn^{-(1-\delta)d}, \quad 1 \le i \le m_n$

for sufficiently large n.

In the following argument we denote by N_n the number of times that $X(I_i)$ visits x, as i increases from 1 to m_n: $N_n := \sum_{i=1}^{m_n}\mathbf{1}_{X(I_i)}(x)$. It follows from (4.8) and the fact that $m_n \le cn^{1/H}$

(4.9) $\mathbb{E}(N_n) \le cn^{1/H}\cdot n^{-(1-\delta)d}.$

Now we consider two different cases.

Case 1. If $dH > 1$, we choose $\delta > 0$ such that $(1-\delta)d > 1/H$. Therefore by (4.9), $\lim_{n\to\infty}\mathbb{E}(N_n) = 0$, which combined with Fatou's lemma yields that $N_n = 0$ for infinitely many n, since the random variables N_n are integer-valued. Therefore $L_x = \varnothing$ a.s..

Case 2. If $dH < 1$, we define a constant $\eta := 1/H - (1-2\delta)d > 0$, where the positivity of η is guaranteed by the fact that $1/H > d$. We consider the sequence of integers $n_j = 2^j$, $j \ge 1$. (4.9) and Chebyshev's inequality imply

$$\mathbb{P}\left(N_{n_j} \ge \epsilon\right) \le \frac{c}{\epsilon}\,2^{j\left(\frac{1}{H}-(1-\delta)d\right)}, \quad \forall \epsilon > 0.$$

Therefore by Borell-Cantelli lemma, it holds that almost surely $N_{n_j} \le cn_j^\eta$ for all j large enough. This implies that $\overline{\dim}_{\mathcal{B}}L_x \le H\eta$ almost surely. Letting $\delta \downarrow 0$ along rational numbers proves

$$\overline{\dim}_{\mathcal{B}}L_x \le 1 - Hd \quad a.s.,$$

which proves the upper bound.   □

The next lemma is on the lower bound of Hausdorff dimension of L_x.

Lemma 4.7. *For every $x \in \mathbb{R}^d$, there exists some $C > 0$ such that*

$$\mathbb{P}\left(\dim_{\mathcal{H}}L_x \ge 1 - Hd\right) \ge C.$$

Proof. To prove the lower bound for $\dim_{\mathcal{H}}L_x$, we fix a small constant $\delta > 0$ such that

$$\gamma := 1 - (1+\delta)Hd > 0.$$

Note that if we can prove that there is a constant $c > 0$ independent of δ such that

(4.10) $\mathbb{P}\{\dim_{\mathcal{H}}L_x \ge \gamma\} \ge c,$

then the lower bound will follow by letting $\delta \downarrow 0$. The strategy to prove (4.10) is standard and based on the capacity argument by Kahane [14]. We spell out all the details for the convenience of the readers.

Let $\mathcal{M}_\gamma^+$ be the space of all non-negative measures on $\mathbb{R}$ with finite γ-energy. It is known that $\mathcal{M}_\gamma^+$ is a complete metric space under the metric $\| \cdot \|_\gamma$ given by

$$\|\mu\|_\gamma^2 = \int_\mathbb{R} \int_\mathbb{R} \frac{\mu(dt)\mu(ds)}{|t-s|^\gamma}.$$

We define a sequence of random positive measures $\mu_n := \mu_n(x, \cdot)$ on the Borel sets C of $[\epsilon, 1]$ by

(4.11)
$$\mu_n(C) = \int_C (2\pi n)^{d/2} \exp\left(-\frac{n|X_t - x|^2}{2}\right) dt.$$

It follows from Kahane [14] or Testard [20] that if there exist positive constants $c_i, 1 \le i \le 3$ such that

(4.12)
$$\mathbb{E}\left(\|\mu_n\|\right) \ge c_1, \quad \mathbb{E}\left(\|\mu_n\|^2\right) \le c_2, \quad \mathbb{E}\left(\|\mu_n\|_\gamma\right) \le c_3 \quad \text{for all } n.,$$

where $\|\mu_n\| = \mu_n([\epsilon, 1])$ denotes the total mass of μ_n, then there is a subsequence of $\{u_n\}$, say $\{\mu_{n_k}\}$, such that $\mu_{n_k} \to \mu \in \mathcal{M}_\gamma^+$, and μ is strictly positive with probability at least $c_1^2/(2c_2)$. It then follows from (4.11) that μ has its support in L_x almost surely. Moreover, the third inequality of (4.11) together with monotone convergence theorem imply that the $\gamma-$energy of μ is finite. Therefore Frostman's theorem yields (4.10) with $c = c_1^2/(2c_2)$. It remains to verify the three inequalities in (4.12).

In the following computation, the strictly positive constant c may change from line to line.

$$\begin{aligned}
\mathbb{E}\left(\|\mu_n\|\right) &= \mathbb{E} \int_{[\epsilon,1]} (2\pi n)^{d/2} \exp\left[-\frac{n|X_t - x|^2}{2}\right] dt \\
&= \int_{[\epsilon,1]} (2\pi n)^{d/2} \mathbb{E} \exp\left[-\frac{n|X_t - x|^2}{2}\right] dt \\
&\ge \int_{[\epsilon,1]} cn^{d/2} \int_{y \in \mathbb{R}^d} e^{-\frac{n|y-x|^2}{2}} p(t, x, y) dy dt \\
&\ge \int_{[\epsilon,1]} cn^{d/2} \int_{|y-x|<1/n} e^{-\frac{n|y-x|^2}{2}} p(t, x, y) dy dt \\
&\ge \int_{[\epsilon,1]} cn^{d/2} \int_{|z-x|<1} p\left(t, x, \frac{z-x}{\sqrt{n}} + x\right) \frac{1}{(\sqrt{n})^d} dz dt > c > 0.
\end{aligned}$$

In the last line above we have performed a change of variable $\sqrt{n}(y - x) = z - x$ and restricted the integration over z in a ball $|z - x| < 1$. Then the desired low bound clearly follows by the fact that $p(t, x, y)$ is globally strictly positive (Theorem 3.5).

For the second inequality of (4.12), by applying the transition density estimate in Theorem 3.4, we get

$$
\mathbb{E}\left(\|\mu_n\|^2\right) = \mathbb{E}\left[\int_{t\in[\epsilon,1]} (2\pi n)^{d/2} \exp\left(-\frac{n|X_t - x|^2}{2}\right) dt \int_{s\in[\epsilon,1]} (2\pi n)^{d/2} \exp\left(-\frac{n|X_s - x|^2}{2}\right) ds\right]
$$

$$
= 2\cdot(2\pi n)^d \cdot \mathbb{E}\left[\int_{t=\epsilon}^1 \int_{s=\epsilon}^t \exp\left(-\frac{n\left(|X_t - x|^2 + |X_s - x|^2\right)}{2}\right) ds\, dt\right]
$$

$$
\le cn^d \int_{z\in\mathbb{R}^d} \int_{y\in\mathbb{R}^d} \int_{t=\epsilon}^1 \int_{s=\epsilon}^1 \exp\left(-\frac{n\left(|y - x|^2 + |z - x|^2\right)}{2}\right) \frac{1}{|t - s|^{Hd}} ds\, dt\, dy\, dz
$$

$$
\le cn^d \int_{z\in\mathbb{R}^d} \int_{y\in\mathbb{R}^d} \int_{t=\epsilon}^1 \int_{s=\epsilon}^1 \frac{1}{|t - s|^{Hd}} \exp\left(-\frac{n\left(|y - x|^2 + |z - x|^2\right)}{2}\right) ds\, dt\, dy\, dz
$$

$$
= cn^d \int_{t\in[\epsilon,1]} \int_{s\in[\epsilon,1]} \frac{1}{|t - s|^{Hd}} ds\, dt \int_{z\in\mathbb{R}^d} e^{-\frac{n|x-z|^2}{4}} dz \int_{y\in\mathbb{R}^d} e^{-\frac{n|y-z|^2}{t}} dy = cn^d \cdot n^{-d/2} \cdot n^{-d/2} \le c,
$$

where the integration in t and s converges because $Hd < 1$.

Similarly, again by Theorem 3.4

$$
\mathbb{E}\|\mu_n\|_\gamma^2 = 2\int_{t=\epsilon}^1 \int_{s=\epsilon}^1 (2\pi n)^d \exp\left[-\frac{n\left(|X_t - x|^2 + |X_s - x|^2\right)}{2}\right] \frac{1}{|t - s|^\gamma} ds\, dt
$$

$$
\le 2\int_{t=\epsilon}^1 dt \int_{s=\epsilon}^1 ds \int_{z\in\mathbb{R}^d} \int_{y\in\mathbb{R}^d} (2\pi n)^d \exp\left[-\frac{n\left(|x - y|^2 + |x - z|^2\right)}{2}\right]
$$

$$
\times \frac{1}{|t - s|^\gamma} \frac{1}{|t - s|^{Hd}} \exp\left\{-\lambda \frac{|y - z|^{2\gamma}}{C|t - s|^{2\gamma^2}}\right\} dy\, dz
$$

$$
\le c\int_{t=\epsilon}^1 \int_{s=\epsilon}^1 \frac{1}{|t - s|^{\gamma+Hd}} ds\, dt,
$$

which converges since $\gamma = 1 - (1 + \delta)Hd$. This proves the third inequality of (4.12). The proof to the upper bound is thus finished. □

The following theorem on the Hausdorff dimension of level sets of X is an immediate consequence of Lemma 4.6 and Lemma 4.7.

Theorem 4.8. *Let X be the solution to (4.2) and $L_x = \{t \in [\epsilon, 1] : X_t = x\}$ be the level set.*
(i) If $dH > 1$, then for every $x \in \mathbb{R}^d$, $L_x = \varnothing$ a.s.
(ii) If $dH < 1$, then for every $x \in \mathbb{R}^d$,

$$
\dim_{\mathcal{H}} L_x = \dim_{\mathcal{P}} L_x = 1 - dH
$$

holds with positive probability.

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